

The bright N -soliton solution of a multi-component modified nonlinear Schrödinger equation

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Abstract

A direct method is developed for constructing the bright N -soliton solution of a multi-component modified nonlinear Schrödinger equation. Specifically, the two different expressions of the solution are obtained both of which are expressed as a rational function of determinants. A simple relation is found between them by employing the properties of the Cauchy matrix. The proof of the solution reduces to the bilinear equations among the bordered determinants in which Jacobi's identity and related formulas play a central role. Last, the bright N -soliton solution is presented for a (2+1)-dimensional nonlocal model equation arising from the multi-component system as the number of dependent variables tends to infinity.

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1. Introduction

The study of multi-component system of nonlinear partial differential equations (PDEs) is current interest in the theory of nonlinear waves. Of particular concern are the multi-component generalizations of the nonlinear Schrödinger (NLS) equation because of their wide applicability in real physical systems such as nonlinear optics, nonlinear water waves and plasma physics [1-3]. The integrable two-component NLS system has been introduced for the first time by Manakov to describe the propagation of polarized electric field in an optical fiber and explored in detail by means of the inverse scattering transform (IST) method [4]. After this remarkable work, various variants of integrable multi-component NLS systems have been proposed and analyzed by the exact method of solution such as the IST and Hirota's direct method. In this paper, we consider the following multi-component system of nonlinear PDEs which is a hybrid of the coupled NLS equation and coupled derivative NLS equation

$$i q_{j,t} + q_{j,xx} + \mu \left(\sum_{k=1}^n |q_k|^2 \right) q_j + i\gamma \left[\left(\sum_{k=1}^n |q_k|^2 \right) q_j \right]_x = 0, \quad (j = 1, 2, \dots, n), \quad (1.1)$$

where $q_j = q_j(x, t)$ ($j = 1, 2, \dots, n$) are complex-valued functions of x and t , μ and γ are real constants, n is an arbitrary positive integer and subscripts x and t appended to q_j denote partial differentiations. The integrability of the above system has been established by constructing the Lax pair and an infinite number of conservation laws [5]. For the two special cases $\gamma = 0$ and $\mu = 0$ reduced from the system (1.1) which correspond to the multi-component NLS and multi-component derivative NLS equations, respectively, their integrability has already been verified in [6, 7]. In the context of plasma physics, the two-component system with $\mu = 0$ is a model equation for the propagation of polarized Alfvén waves. The single bright soliton solution to this system has been obtained by means of the IST [8]. The two-component system with $\mu \neq 0$ and $\gamma \neq 0$ has been derived as a model for describing the propagation of ultra-short pulses in birefringent optical fibers, together with its soliton solutions [9]. Quite recently, we obtained the general bright multisoliton solution (i.e., the N -soliton solution with N being an arbitrary positive integer which vanishes at infinity) of the two-component system by using a direct method [10]. It is

important to remark that the constant μ must be nonnegative to support smooth bright solitons. The negative case is worth studying as well which will be treated in a separate issue.

The purpose of this paper is to extend the results obtained in [10] to the general n -component system. Specifically, we present the bright N -soliton solution of the n -component system (1.1) in the form of compact determinantal expressions. Although the construction of the solution can be done following the similar procedure to that developed in [10] for the two-component system, we provide a novel proof of the solution using the expansion formulas for the bordered determinant.

This paper is organized as follows. In section 2, we first transform the system (1.1) to a gauge equivalent system and then recast it to a system of bilinear equations by introducing appropriate dependent variable transformations. Subsequently, the bright N -soliton solution to the bilinear equations is presented. It has a simple structure expressed in terms of certain determinants. In section 3, we introduce some notations associated with the bright N -soliton solution and then prove several key formulas for determinants. In section 4, we perform the proof of the bright N -soliton solution using an elementary theory of determinants in which Jacobi's identity and related formulas will play a central role. In section 5, we provide an alternative expression of the bright N -soliton solution and compare it with the corresponding solution presented in section 2. We then find a simple relation between two types of solutions by employing the properties of the Cauchy matrix. This result leads to an alternative proof of the solution in a very simple way. In section 6, we discuss a $(2+1)$ -dimensional nonlocal modified NLS equation arising from the continuum limit $n \rightarrow \infty$ of the system (1.1). Specifically, we demonstrate that its bright N -soliton solution can be generated simply from that of the n -component system. Section 7 is devoted to concluding remarks where we will comment on existing literatures about the bright N -soliton solutions of the multi-component NLS equation (equation (1.1) with $\gamma = 0$) and show that the solutions obtained in this paper include these known solutions as special cases.

2. Bilinearization and bright N -soliton solution

2.1. Bilinearization

We first apply the gauge transformations

$$q_j = u_j \exp \left[-\frac{i\gamma}{2} \int_{-\infty}^x \sum_{k=1}^n |u_k|^2 dx \right], \quad (j = 1, 2, \dots, n), \quad (2.1)$$

to the system (1.1) subjected to the the boundary conditions $q_j \rightarrow 0, u_j \rightarrow 0$ ($j = 1, 2, \dots, n$) as $|x| \rightarrow \infty$, where $u_j = u_j(x, t)$ ($j = 1, 2, \dots, n$) are complex-valued functions of x and t . Then, we obtain the system of nonlinear PDEs for u_j

$$i u_{j,t} + u_{j,xx} + \mu \left(\sum_{k=1}^n |u_k|^2 \right) u_j + i\gamma \left(\sum_{k=1}^n u_k^* u_{k,x} \right) u_j = 0, \quad (j = 1, 2, \dots, n), \quad (2.2)$$

where the asterisk appended to u_k denotes complex conjugate. This notation will be used frequently hereafter. The second step in our analysis is given by the following proposition:

Proposition 2.1. *By means of the dependent variable transformations*

$$u_j = \frac{g_j}{f}, \quad (j = 1, 2, \dots, n), \quad (2.3)$$

the system of nonlinear PDEs (2.2) can be decoupled into the following system of bilinear equations for f and g_j

$$(iD_t + D_x^2)g_j \cdot f = 0, \quad (j = 1, 2, \dots, n), \quad (2.4)$$

$$D_x f \cdot f^* = \frac{i\gamma}{2} \sum_{k=1}^n |g_k|^2, \quad (2.5)$$

$$D_x^2 f \cdot f^* = \mu \sum_{k=1}^n |g_k|^2 + \frac{i\gamma}{2} \sum_{k=1}^n D_x g_k \cdot g_k^*. \quad (2.6)$$

Here, $f = f(x, t)$ and $g_j = g_j(x, t)$ ($j = 1, 2, \dots, n$) are complex-valued functions of x and t and the bilinear operators D_x and D_t are defined by

$$D_x^m D_t^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x, t) g(x', t') \Big|_{x'=x, t'=t}, \quad (2.7)$$

where m and n are nonnegative integers.

Proof. Substituting (2.3) into (2.2) and rewriting the resultant equation in terms of the bilinear operators, equations (2.2) can be rewritten as

$$\frac{1}{f^2}(\mathrm{i}D_t g_j \cdot f + D_x^2 g_j \cdot f) + \frac{g_j}{f^3 f^*} \left(-f^* D_x^2 f \cdot f + \mu f \sum_{k=1}^n |g_k|^2 + \mathrm{i}\gamma \sum_{k=1}^n g_k^* D_x g_k \cdot f \right) = 0, \\ (j = 1, 2, \dots, n). \quad (2.8)$$

Insert the identity

$$f^* D_x^2 f \cdot f = f D_x^2 f \cdot f^* - 2f_x D_x f \cdot f^* + f(D_x f \cdot f^*)_x, \quad (2.9)$$

into the second term on the left-hand side of (2.8). Then, equations (2.8) become

$$\frac{1}{f^2}(\mathrm{i}D_t g_j \cdot f + D_x^2 g_j \cdot f) + \frac{g_j}{f^3 f^*} \left[f \left\{ -D_x^2 f \cdot f^* + \mu \sum_{k=1}^n |g_k|^2 - (D_x f \cdot f^*)_x + \mathrm{i}\gamma \sum_{k=1}^n g_k^* g_{k,x} \right\} \right. \\ \left. + f_x \left\{ 2D_x f \cdot f^* - \mathrm{i}\gamma \sum_{k=1}^n |g_k|^2 \right\} \right] = 0, \quad (j = 1, 2, \dots, n). \quad (2.10)$$

As easily confirmed by a direct calculation, the left-hand side of (2.10) becomes zero by virtue of equations (2.4)-(2.6). \square

It now follows from (2.3) and (2.5) that

$$-\frac{\mathrm{i}\gamma}{2} \sum_{k=1}^n |u_k|^2 = \frac{\partial}{\partial x} \ln \frac{f^*}{f}, \quad (2.11)$$

which, substituted into (2.1), yields the solution of the system (1.1) in the form

$$q_j = \frac{g_j f^*}{f^2}, \quad (j = 1, 2, \dots, n). \quad (2.12)$$

Note that for the n -component NLS equation (the system (1.1) with $\gamma = 0$), the solution (2.12) simplifies to $q_j = g_j/f$. Indeed, if $\gamma = 0$, then the bilinear equation (2.5) reduces to $D_x f \cdot f^* = 0$. Thus, the ratio f^*/f turns out to be an arbitrary function of t which can be set to 1 under appropriate boundary condition.

2.2. Bright N -soliton solution

We now state the main result in this paper:

Theorem 2.1. *The bright N -soliton solution of the system of bilinear equations (2.4)-(2.6) is given by the determinants f and g_j ($j = 1, 2, \dots, n$) where*

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g_j = \begin{vmatrix} A & I & \mathbf{z}^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{a}_j^* & 0 \end{vmatrix}, \quad (j = 1, 2, \dots, n). \quad (2.13)$$

Here, A, B and I are $N \times N$ matrices and \mathbf{z}, \mathbf{a}_j and $\mathbf{0}$ are N -component row vectors defined below and the symbol T denotes the transpose:

$$A = (a_{jk})_{1 \leq j, k \leq N}, \quad a_{jk} = \frac{1}{2} \frac{z_j z_k^*}{p_j + p_k^*}, \quad z_j = \exp(p_j x + i p_j^2 t), \quad (2.14a)$$

$$B = (b_{jk})_{1 \leq j, k \leq N}, \quad b_{jk} = \frac{(\mu + i \gamma p_k) c_{jk}}{p_j^* + p_k}, \quad c_{jk} = \sum_{s=1}^n \alpha_{sj} \alpha_{sk}^*, \quad (2.14b)$$

$$I = (\delta_{jk})_{1 \leq j, k \leq N}, \quad N \times N \text{ unit matrix}, \quad (2.14c)$$

$$\mathbf{z} = (z_1, z_2, \dots, z_N), \quad \mathbf{a}_j = (\alpha_{j1}, \alpha_{j2}, \dots, \alpha_{jN}), \quad \mathbf{0} = (0, 0, \dots, 0). \quad (2.14d)$$

The above bright N -soliton solution is characterized by $(n+1)N$ complex parameters p_j ($j = 1, 2, \dots, n$) and α_{sj} ($s = 1, 2, \dots, n; j = 1, 2, \dots, N$). The former parameters determine the amplitude and velocity of the solitons whereas the latter ones determine the polarizations and the envelope phases of the solitons. The conditions $p_j + p_k^* \neq 0$ for all j and k and $p_j \neq p_k$ for $j \neq k$ may be imposed to assure the regularity of the solution. In the special case of $n = 2$, (2.13) and (2.14) reproduce the bright N -soliton solution presented in [10]. The proof of Theorem 2.1 will be given in section 4.

To simplify the proof of theorem 2.1, the following observation is useful:

Proposition 2.2. *If we introduce the gauge transformations*

$$f = \tilde{f}, \quad g_j = \exp \left[i \left\{ \frac{\mu}{\gamma} \tilde{x} + \left(\frac{\mu}{\gamma} \right)^2 \tilde{t} \right\} \right] \tilde{g}_j, \quad (j = 1, 2, \dots, n), \quad (2.15a)$$

$$x = \tilde{x} + \frac{2\mu}{\gamma} \tilde{t}, \quad t = \tilde{t}, \quad (2.15b)$$

then the bilinear equations (2.4)-(2.6) recast to

$$(\mathrm{i}D_{\tilde{t}} + D_{\tilde{x}}^2)\tilde{g}_j \cdot \tilde{f} = 0, \quad (j = 1, 2, \dots, n), \quad (2.16)$$

$$D_{\tilde{x}}\tilde{f} \cdot \tilde{f}^* = \frac{\mathrm{i}\gamma}{2} \sum_{k=1}^n |\tilde{g}_k|^2, \quad (2.17)$$

$$D_{\tilde{x}}^2\tilde{f} \cdot \tilde{f}^* = \frac{\mathrm{i}\gamma}{2} \sum_{k=1}^n D_{\tilde{x}}\tilde{g}_k \cdot \tilde{g}_k^*, \quad (2.18)$$

respectively.

Proof. The proof can be done by a straightforward calculation. \square

Thus, the form of equations (2.4) and (2.5) is unchanged whereas equation (2.6) becomes a simplified equation with $\mu = 0$. Consequently, the proof of the N -soliton solution may be performed for the corresponding solution with $\mu = 0$. Hence, in the analysis developed in the following sections, we put $\mu = 0$ without loss of generality.

3. Notation and some basic formulas for determinants

In this section, we first introduce the notation for matrices and then provide some basic formulas for determinants.

3.1. Notation

We define the following matrices associated with the N -soliton solution (2.13) with (2.14):

$$D = \begin{pmatrix} A & I \\ -I & B \end{pmatrix}, \quad (3.1)$$

$$D(\mathbf{a}^*; \mathbf{b}) = \begin{pmatrix} A & I & \mathbf{0}^T \\ -I & B & \mathbf{b}^T \\ \mathbf{0} & \mathbf{a}^* & 0 \end{pmatrix}, \quad D(\mathbf{a}^*; \mathbf{z}) = \begin{pmatrix} A & I & \mathbf{z}^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & \mathbf{a}^* & 0 \end{pmatrix}, \quad D(\mathbf{z}^*; \mathbf{z}) = \begin{pmatrix} A & I & \mathbf{z}^T \\ -I & B & \mathbf{0}^T \\ \mathbf{z}^* & \mathbf{0} & 0 \end{pmatrix}. \quad (3.2)$$

Note the position of the vectors \mathbf{a}^* , \mathbf{b} , \mathbf{z} and \mathbf{z}^* in the above expressions. The matrices which include more than two vectors will be introduced as well. For example,

$$D(\mathbf{a}^*, \mathbf{z}^*; \mathbf{b}, \mathbf{z}) = \begin{pmatrix} A & I & \mathbf{0} & \mathbf{z}^T \\ -I & B & \mathbf{b}^T & \mathbf{0}^T \\ \mathbf{0} & \mathbf{a}^* & 0 & 0 \\ \mathbf{z}^* & \mathbf{0} & 0 & 0 \end{pmatrix}, \quad D(\mathbf{a}^*, \mathbf{z}^*; \mathbf{z}, \mathbf{z}') = \begin{pmatrix} A & I & \mathbf{z}^T & \mathbf{z}'^T \\ -I & B & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & \mathbf{a}^* & 0 & 0 \\ \mathbf{z}^* & \mathbf{0} & 0 & 0 \end{pmatrix}. \quad (3.3)$$

3.2. Formulas for determinants

Let $A = (a_{jk})_{1 \leq j, k \leq M}$ be an $M \times M$ matrix with M being an arbitrary positive integer, A_{jk} be the cofactor of the element a_{jk} and $\mathbf{a}, \mathbf{b}, \mathbf{a}_j$ and \mathbf{b}_j ($j = 1, 2, \dots, n$) be M -component row vectors. The following well-known formulas are used frequently in our analysis [11]:

$$\frac{\partial}{\partial x}|A| = \sum_{j,k=1}^M \frac{\partial a_{jk}}{\partial x} A_{jk}, \quad (3.4)$$

$$\begin{vmatrix} A & \mathbf{a}^T \\ \mathbf{b} & z \end{vmatrix} = |A|z - \sum_{j,k=1}^M A_{jk} a_j b_k, \quad (3.5)$$

$$|A(\mathbf{a}_1, \mathbf{a}_2; \mathbf{b}_1, \mathbf{b}_2)||A| = |A(\mathbf{a}_1; \mathbf{b}_1)||A(\mathbf{a}_2; \mathbf{b}_2)| - |A(\mathbf{a}_1; \mathbf{b}_2)||A(\mathbf{a}_2; \mathbf{b}_1)|. \quad (3.6)$$

The formula (3.4) is the differentiation rule of the determinant and (3.5) is the expansion formula for a bordered determinant with respect to the last row and last column. The formula (3.6) is Jacobi's identity.

The following two formulas may not be well-known but are very important in our analysis. In particular, formula (3.7) gives rise to the expansion formulas for the bordered determinant (see (3.9) and (3.10) below):

$$|A(\mathbf{a}_1, \dots, \mathbf{a}_n; \mathbf{b}_1, \dots, \mathbf{b}_n)||A|^{n-1} = \begin{vmatrix} |A(\mathbf{a}_1; \mathbf{b}_1)| & \cdots & |A(\mathbf{a}_1; \mathbf{b}_n)| \\ \vdots & \ddots & \vdots \\ |A(\mathbf{a}_n; \mathbf{b}_1)| & \cdots & |A(\mathbf{a}_n; \mathbf{b}_n)| \end{vmatrix}, \quad (n \geq 2), \quad (3.7)$$

$$\begin{aligned} \left| A + \epsilon \sum_{s=1}^n \mathbf{b}_s^T \mathbf{a}_s \right| &= |A| + \sum_{m=1}^{n'} (-\epsilon)^m \sum_{1 \leq s_1 < \dots < s_m \leq n} |A(\mathbf{a}_{s_1}, \dots, \mathbf{a}_{s_m}; \mathbf{b}_{s_1}, \dots, \mathbf{b}_{s_m})| \\ &= |A| + \sum_{m=1}^{n'} \frac{(-\epsilon)^m}{m!} \sum_{s_1, \dots, s_m=1}^n |A(\mathbf{a}_{s_1}, \dots, \mathbf{a}_{s_m}; \mathbf{b}_{s_1}, \dots, \mathbf{b}_{s_m})|. \end{aligned} \quad (3.8)$$

Here, ϵ is an arbitrary parameter, the notation $\mathbf{b}_s^T \mathbf{a}_s$ on the left-hand side of (3.8) represents an $M \times M$ matrix whose (j, k) element is given by $\beta_{sj} \alpha_{sk}$ and $n' = \min(n, M)$. The formula (3.7) is a variant of the Sylvester theorem in the theory of determinants.

Proof of (3.7). The proof proceeds by a mathematical induction. For $n = 2$, (3.7) reduces to Jacobi's identity (3.6). Assume that formula (3.7) is true for $n - 1$ ($n \geq 3$).

Let L be the left-hand side of (3.7). Recall that the determinant changes its sign if any two rows (or columns) are interchanged. Applying this rule repeatedly to L when \mathbf{a}_n and \mathbf{b}_n are shifted in front of \mathbf{a}_1 and \mathbf{b}_1 , respectively,

$$L = |\hat{A}(\mathbf{a}_1, \dots, \mathbf{a}_{n-1}; \mathbf{b}_1, \dots, \mathbf{b}_{n-1})| |A|^{n-1},$$

where $\hat{A} = A(\mathbf{a}_n; \mathbf{b}_n)$ is an $(M+1) \times (M+1)$ matrix which is assumed to be nonsingular, i.e., $|\hat{A}| \neq 0$. In view of the inductive hypothesis, L can be written as

$$L = \frac{|A|^{n-1}}{|\hat{A}|^{n-2}} \begin{vmatrix} |\hat{A}(\mathbf{a}_1; \mathbf{b}_1)| & \cdots & |\hat{A}(\mathbf{a}_1; \mathbf{b}_{n-1})| \\ \vdots & \ddots & \vdots \\ |\hat{A}(\mathbf{a}_{n-1}; \mathbf{b}_1)| & \cdots & |\hat{A}(\mathbf{a}_{n-1}; \mathbf{b}_{n-1})| \end{vmatrix}.$$

It follows from Jacobi's identity (3.6) that

$$|\hat{A}(\mathbf{a}_j; \mathbf{b}_k)| |A| = |A(\mathbf{a}_n, \mathbf{a}_j; \mathbf{b}_n, \mathbf{b}_k)| |A| = |A(\mathbf{a}_n; \mathbf{b}_n)| |A(\mathbf{a}_j; \mathbf{b}_k)| - |A(\mathbf{a}_j; \mathbf{b}_n)| |A(\mathbf{a}_n; \mathbf{b}_k)|.$$

which, substituted into L , recasts L into the form

$$L = \frac{1}{|A(\mathbf{a}_n; \mathbf{b}_n)|^{n-2}} \left| \left(|A(\mathbf{a}_n; \mathbf{b}_n)| |A(\mathbf{a}_j; \mathbf{b}_k)| - |A(\mathbf{a}_j; \mathbf{b}_n)| |A(\mathbf{a}_n; \mathbf{b}_k)| \right)_{1 \leq j, k \leq n-1} \right|.$$

Referring to the property of the bordered determinant, the above expression simplifies to

$$L = \frac{1}{|A(\mathbf{a}_n; \mathbf{b}_n)|^{n-2}} \begin{vmatrix} |A(\mathbf{a}_n; \mathbf{b}_n)| |A(\mathbf{a}_1; \mathbf{b}_1)| & \cdots & |A(\mathbf{a}_n; \mathbf{b}_n)| |A(\mathbf{a}_1; \mathbf{b}_{n-1})| & |A(\mathbf{a}_1; \mathbf{b}_n)| \\ \vdots & \ddots & \vdots & \vdots \\ |A(\mathbf{a}_n; \mathbf{b}_n)| |A(\mathbf{a}_{n-1}; \mathbf{b}_1)| & \cdots & |A(\mathbf{a}_n; \mathbf{b}_n)| |A(\mathbf{a}_{n-1}; \mathbf{b}_{n-1})| & |A(\mathbf{a}_{n-1}; \mathbf{b}_n)| \\ |A(\mathbf{a}_n; \mathbf{b}_1)| & \cdots & |A(\mathbf{a}_n; \mathbf{b}_{n-1})| & 1 \end{vmatrix}$$

Extract the factor $|A(\mathbf{a}_n; \mathbf{b}_n)|$ from the j th row ($j = 1, 2, \dots, n-1$) and then multiply the last column by the same factor. Then, the resultant expression is seen to be equal to the right-hand side of (3.7). \square

Proof of (3.8) For $n = 1$, it follows by using the property of the bordered determinant that

$$\begin{aligned} |A + \epsilon \mathbf{b}_1^T \mathbf{a}_1| &= \begin{vmatrix} a_{11} + \epsilon \beta_{11} \alpha_{11} & \cdots & a_{1M} + \epsilon \beta_{11} \alpha_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} + \epsilon \beta_{1M} \alpha_{11} & \cdots & a_{MM} + \epsilon \beta_{1M} \alpha_{1M} \end{vmatrix} \\ &= \epsilon \begin{vmatrix} a_{11} & \cdots & a_{1M} & \beta_{11} \\ \vdots & \ddots & \vdots & \vdots \\ a_{M1} & \cdots & a_{MM} & \beta_{1M} \\ -\alpha_{11} & \cdots & -\alpha_{1M} & \epsilon^{-1} \end{vmatrix} = \epsilon \begin{vmatrix} A & \mathbf{b}_1^T \\ -\mathbf{a}_1 & \epsilon^{-1} \end{vmatrix}. \end{aligned}$$

Repeated use of the above modification yields

$$\left| A + \epsilon \sum_{s=1}^n \mathbf{b}_s^T \mathbf{a}_s \right| = \epsilon^n \begin{vmatrix} A & \mathbf{b}_1^T & \cdots & \mathbf{b}_n^T \\ -\mathbf{a}_1 & \epsilon^{-1} & \cdots & 0 \\ \vdots & \mathbf{0}^T & \ddots & \mathbf{0}^T \\ -\mathbf{a}_n & 0 & \mathbf{0} & \epsilon^{-1} \end{vmatrix}.$$

Expanding the determinant in powers of ϵ^{-1} , it is found that

$$\left| A + \epsilon \sum_{s=1}^n \mathbf{b}_s^T \mathbf{a}_s \right| = |A| + \epsilon^n \sum_{m=1}^n (-1)^m \sum_{1 \leq s_1 < \cdots < s_m \leq n} |A(\mathbf{a}_{s_1}, \dots, \mathbf{a}_{s_m}; \mathbf{b}_{s_1}, \dots, \mathbf{b}_{s_m})| \epsilon^{-(n-m)}.$$

The above expression coincides with (3.8) for $n \leq M$ since $n' = \min(n, M) = n$. For $M+1 \leq n$, on the other hand, the determinant $|A(\mathbf{a}_{s_1}, \dots, \mathbf{a}_{s_m}; \mathbf{b}_{s_1}, \dots, \mathbf{b}_{s_m})|$ becomes zero identically for $M+1 \leq m \leq n$, as confirmed by the Laplace expansion of the determinant with respect to the last m rows, for example. This implies that the summation with respect to m is truncated at $m = M$ which is in accordance with (3.8) since $n' = \min(n, M) = M$. The second line of (3.8) follows from the facts that any permutation of the indices $\{s_1, s_2, \dots, s_m\}$ does not alter the value of the determinant $|A(\mathbf{a}_{s_1}, \dots, \mathbf{a}_{s_m}; \mathbf{b}_{s_1}, \dots, \mathbf{b}_{s_m})|$ and the total number of the permutation is $m!$, and if the determinant includes at least two same rows (or columns), then it becomes zero identically. \square

Suppose that $|A| \neq 0$. Expanding the determinant on the right-hand side of (3.7) with respect to the first column and using (3.7) with n replaced by $n-1$, we then obtain an expansion formula

$$|A(\mathbf{a}_1, \dots, \mathbf{a}_n; \mathbf{b}_1, \dots, \mathbf{b}_n)| = \frac{1}{|A|} \sum_{j=1}^n (-1)^{j-1} |A(\mathbf{a}_j, \mathbf{b}_1)| |A(\mathbf{a}_1, \dots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \dots, \mathbf{a}_n; \mathbf{b}_2, \dots, \mathbf{b}_n)|. \quad (3.9)$$

Similarly, the expansion with respect to the first row gives

$$|A(\mathbf{a}_1, \dots, \mathbf{a}_n; \mathbf{b}_1, \dots, \mathbf{b}_n)| = \frac{1}{|A|} \sum_{j=1}^n (-1)^{j-1} |A(\mathbf{a}_1, \mathbf{b}_j)| |A(\mathbf{a}_2, \dots, \mathbf{a}_n; \mathbf{b}_1, \dots, \mathbf{b}_{j-1}, \mathbf{b}_{j+1}, \dots, \mathbf{b}_n)|. \quad (3.10)$$

The above two formulas will be used effectively to prove the bilinear equations (2.5) and (2.6).

4. Proof of the bright N -soliton solution

In this section, we show that the bright N -soliton solution (2.13) with (2.14) satisfies the system of bilinear equations (2.4)-(2.6). The proof can be performed for $\mu = 0$, as noted at the end of section 2. We first prove some formulas associated with the determinants f and g_j ($j = 1, 2, \dots, n$) and then proceed to the proof.

4.1. Formulas

In terms of the notation introduced in section 3.1 (see (3.1) and (3.2)), f and g_j are written in the form

$$f = |D|, \quad g_j = -|D(\mathbf{a}_j^*; \mathbf{z})|, \quad (j = 1, 2, \dots, n). \quad (4.1)$$

The differentiation rules of f and g_j with respect to t and x are given by the following formulas:

Lemma 4.1.

$$f_t = -\frac{i}{2} \{ |D(\mathbf{z}^*; \mathbf{z}_x)| - |D(\mathbf{z}_x^*; \mathbf{z})| \}, \quad (4.2)$$

$$f_x = -\frac{1}{2} |D(\mathbf{z}^*; \mathbf{z})|, \quad (4.3)$$

$$f_{xx} = -\frac{1}{2} \{ |D(\mathbf{z}^*; \mathbf{z}_x)| + |D(\mathbf{z}_x^*; \mathbf{z})| \}, \quad (4.4)$$

$$g_{j,t} = -|D(\mathbf{a}_j^*; \mathbf{z}_t)| + \frac{i}{2} |D(\mathbf{a}_j^*, \mathbf{z}^*; \mathbf{z}, \mathbf{z}_x)|, \quad (4.5)$$

$$g_{j,x} = -|D(\mathbf{a}_j^*; \mathbf{z}_x)|, \quad (4.6)$$

$$g_{j,xx} = -|D(\mathbf{a}_j^*; \mathbf{z}_{xx})| + \frac{1}{2} |D(\mathbf{a}_j^*, \mathbf{z}^*; \mathbf{z}_x, \mathbf{z})|. \quad (4.7)$$

Here, \mathbf{z}_t , \mathbf{z}_x and \mathbf{z}_{xx} are N -component row vectors given by $\mathbf{z}_t = (ip_1^2 z_1, ip_2^2 z_2, \dots, ip_N^2 z_N)$, $\mathbf{z}_x = (p_1 z_1, p_2 z_2, \dots, p_N z_N)$ and $\mathbf{z}_{xx} = (p_1^2 z_1, p_2^2 z_2, \dots, p_N^2 z_N)$, respectively.

Proof. We prove (4.2). Let $D = (d_{jk})_{1 \leq j, k \leq 2N}$ be a $2N \times 2N$ matrix and D_{jk} be the cofactor of the element d_{jk} . It follows by applying the formula (3.4) to the determinant

f given by (2.13) that

$$\begin{aligned} f_t &= \frac{i}{2} \sum_{j,k=1}^N D_{jk} (p_j - p_k^*) z_j z_k^* \\ &= \frac{i}{2} \sum_{j,k=1}^N D_{jk} (z_{j,x} z_k^* - z_j z_{k,x}^*), \end{aligned}$$

where in passing to the second line, use has been made of the relations $p_j z_j = z_{j,x}$, $p_k^* z_k^* = z_{k,x}^*$. Referring to the formula (3.5) with $z = 0$ and taking into account the notation (3.2), the above expression reduces to the right-hand side of (4.2). A key feature in the proof is that the factor $(p_j + p_k^*)^{-1}$ in the element a_{jk} has been canceled after differentiation with respect to t . Using formulas (3.4) and (3.5) as well as some basic properties of determinants, formulas (4.3)-(4.7) can be proved in the same way. \square

The complex conjugate expressions of f, f_x and g_j can be expressed as follows:

Lemma 4.2.

$$f^* = |\bar{D}|, \quad \bar{D} \equiv \begin{pmatrix} A & I \\ -I & B - i\gamma C \end{pmatrix}, \quad (4.8)$$

$$f_x^* = -\frac{1}{2} |\bar{D}(\mathbf{z}^*; \mathbf{z})|, \quad (4.9)$$

$$g_j^* = |\bar{D}(\mathbf{z}^*; \mathbf{a}_j)|. \quad (4.10)$$

Proof. We prove (4.8). It follows from (2.14a) and (2.14b) that $A^* = A^T$ and $B^* = B^T - i\gamma C^T$ where C is an $N \times N$ matrix with elements c_{jk} defined by (2.14b). These relations lead to the expression of f^*

$$f^* = \begin{vmatrix} A^* & I \\ -I & B^* \end{vmatrix} = \begin{vmatrix} A^T & I \\ -I & (B - i\gamma C)^T \end{vmatrix}.$$

Since $|A^T| = |A|$ for any square matrix A , the above expression reduces to to the right-hand side of (4.8) after multiplying the j th row and k th column ($j, k = 1, 2, \dots, N$) by a factor -1 . Differentiating (4.8) with respect to x and applying formula (3.5) with $z = 0$ to the

resulting expression, formula (4.9) follows immediately. The proof of formula (4.10) can be done in the same way. \square

The following formulas will be used in the proof of (2.5) and (2.6):

Lemma 4.3.

$$|\bar{D}| = |D| + \frac{1}{2}|D(\mathbf{z}^*; \tilde{\mathbf{z}})|, \quad (4.11)$$

$$|D(\mathbf{b}_k^*; \tilde{\mathbf{z}})| = |\bar{D}(\mathbf{a}_k^*; \mathbf{z})|, \quad (4.12)$$

$$|\bar{D}(\mathbf{a}_k^*; \mathbf{b}_k)| = -|D(\mathbf{b}_k^*; \mathbf{a}_k)| - \frac{1}{2}|D(\mathbf{b}_k^*, \mathbf{z}^*; \mathbf{a}_k, \tilde{\mathbf{z}})|, \quad (4.13)$$

$$|\bar{D}(\mathbf{a}_k^*; \mathbf{z}_x)| = |D(\mathbf{b}_k^*; \mathbf{z}) + \frac{1}{2}|D(\mathbf{b}_k^*, \mathbf{z}^*; \mathbf{z}, \tilde{\mathbf{z}})|. \quad (4.14)$$

$$|D(\mathbf{z}^*; \mathbf{z})| = 2i\gamma \sum_{k=1}^n |D(\mathbf{b}_k^*; \mathbf{a}_k)|, \quad (4.15)$$

$$|\bar{D}(\mathbf{z}^*; \mathbf{z})| = -2i\gamma \sum_{k=1}^n |\bar{D}(\mathbf{a}_k^*; \mathbf{b}_k)|, \quad (4.16)$$

where $\tilde{\mathbf{z}}$ and \mathbf{b}_k are N -component row vectors given respectively by $\tilde{\mathbf{z}} = (z_1/p_1, z_2/p_2, \dots, z_N/p_N)$ and $\mathbf{b}_k = (\alpha_{k1}p_1^*, \alpha_{k2}p_2^*, \dots, \alpha_{kN}p_N^*)$.

Proof. First, we prove (4.11). A direct calculation using the elements of B and C given by (2.14b) reveals that

$$b_{jk} - i\gamma c_{jk} = -\frac{p_j^*}{p_k} b_{jk}.$$

The determinant $|\bar{D}|$ from (4.8) is now modified to the form

$$|\bar{D}| = \begin{vmatrix} A & I \\ -I & \left(-\frac{p_j^*}{p_k} b_{jk}\right) \end{vmatrix} = \begin{vmatrix} \left(-\frac{p_k^*}{p_j} a_{jk}\right) & I \\ -I & B \end{vmatrix},$$

where the last line of the above expression follows immediately from the property of the determinant. The definition (2.14a) of a_{jk} now gives

$$-\frac{p_k^*}{p_j} a_{jk} = a_{jk} - \frac{z_j z_k^*}{2p_j}.$$

In view of the property of the bordered determinant, $|\bar{D}|$ is modified in the form

$$|\bar{D}| = \begin{vmatrix} A & I & \tilde{\mathbf{z}}^T \\ -I & B & \mathbf{0}^T \\ \frac{\mathbf{z}^*}{2} & \mathbf{0} & 1 \end{vmatrix},$$

which is seen to coincide with (4.11) by applying formula (3.5). The proof of (4.12)-(4.14) can be done in the same way. Hence, we omit the proof.

Let us now proceed to the proof of (4.15). To this end, it is to be noted that the determinant f can be rewritten in the form

$$f = \begin{vmatrix} \tilde{A} & I \\ -I & \tilde{B} \end{vmatrix},$$

where \tilde{A} and \tilde{B} are $N \times N$ matrices defined by

$$\begin{aligned} \tilde{A} &= (\tilde{a}_{jk})_{1 \leq j, k \leq N}, \quad \tilde{a}_{jk} = \frac{1}{2} \frac{1}{p_j + p_k^*}, \\ \tilde{B} &= (\tilde{b}_{jk})_{1 \leq j, k \leq N}, \quad \tilde{b}_{jk} = \frac{i\gamma c_{jk} p_k}{p_j^* + p_k} z_j^* z_k. \end{aligned}$$

Indeed, the above expression of f is derived from f from (4.1) by extracting the factors z_j and z_k^* from the j th row and k th column, respectively for $j, k = 1, 2, \dots, N$ and then multiplying the $(N+j)$ th row and $(N+k)$ th column by the factors z_j^* and z_k , respectively for $j, k = 1, 2, \dots, N$. Using the formulas (3.4) and (3.5) gives an alternative expression of f_x

$$f_x = -i\gamma \sum_{k=1}^n |D(\mathbf{b}_k^*; \mathbf{a}_k)|.$$

The formula (4.15) follows by comparing this expression with (4.3). The formula (4.16) comes from the complex conjugate expression of (4.15). \square

4.2. Proof of (2.4)

The proof of (2.4) proceeds following the same procedure as that of the same equation for $n = 2$ [10]. Let P_1 be

$$P_1 = (iD_t + D_x^2)g_j \cdot f. \tag{4.17}$$

Substituting (4.1)-(4.7) into (4.17), P_1 becomes

$$P_1 = -|D(\mathbf{a}_j^*, \mathbf{z}^*; \mathbf{z}, \mathbf{z}_x)| |D| + |D(\mathbf{a}_j^*; \mathbf{z})| |D(\mathbf{z}^*; \mathbf{z}_x)| - |D(\mathbf{a}_j^*; \mathbf{z}_x)| |D(\mathbf{z}^*; \mathbf{z})|$$

$$- \{i|D(\mathbf{a}_j^*; \mathbf{z}_t)| + |D(\mathbf{a}_j^*; \mathbf{z}_{xx})|\}. \quad (4.18)$$

Referring to Jacobi's identity (3.6) and the fundamental formula $\alpha|D(\mathbf{a}; \mathbf{b}_1)| + \beta|D(\mathbf{a}; \mathbf{b}_2)| = |D(\mathbf{a}; \alpha\mathbf{b}_1 + \beta\mathbf{b}_2)|$ ($\alpha, \beta \in \mathbb{C}$), P_1 simplifies to $P_1 = -|D(\mathbf{a}_j^*; i\mathbf{z}_t + \mathbf{z}_{xx})|$. Since $i\mathbf{z}_t + \mathbf{z}_{xx} = \mathbf{0}$ by (2.14a), the last column of the determinant consists only of zero elements, implying that $P_1 = 0$. \square

4.3. Proof of (2.5)

The equation to be proved is $P_2 = 0$, where

$$P_2 = D_x f \cdot f^* - \frac{i\gamma}{2} \sum_{k=1}^n |g_k|^2. \quad (4.19)$$

Substituting (4.1), (4.3) and (4.8)-(4.10) into (4.19), P_2 becomes

$$P_2 = -\frac{1}{2}|\bar{D}||D(\mathbf{z}^*; \mathbf{z})| + \frac{1}{2}|D||\bar{D}(\mathbf{z}^*; \mathbf{z})| + \frac{i\gamma}{2} \sum_{k=1}^n |D(\mathbf{a}_k^*; \mathbf{z})||\bar{D}(\mathbf{z}^*; \mathbf{a}_k)|. \quad (4.20)$$

Further simplification is possible with use of (4.11), (4.15) and (4.16) with (4.13), giving rise to

$$P_2 = \frac{i\gamma}{2} \sum_{k=1}^n \left(-|D(\mathbf{b}_k^*; \mathbf{a}_k)||D(\mathbf{z}^*; \tilde{\mathbf{z}})| + |D(\mathbf{b}_k^*; \mathbf{z}^*; \mathbf{a}_k, \tilde{\mathbf{z}})||D| + |D(\mathbf{a}_k^*; \mathbf{z})||\bar{D}(\mathbf{z}^*; \mathbf{a}_k)| \right). \quad (4.21)$$

Applying Jacobi's identity (3.6) to the middle term and replacing $|D(\mathbf{b}_k^*; \tilde{\mathbf{z}})|$ by the right-hand side of (4.12) in the resultant expression, P_2 reduces to

$$P_2 = \frac{i\gamma}{2} \sum_{k=1}^n \left(-|\bar{D}(\mathbf{a}_k^*; \mathbf{z})||D(\mathbf{z}^*; \mathbf{a}_k)| + |D(\mathbf{a}_k^*; \mathbf{z})||\bar{D}(\mathbf{z}^*; \mathbf{a}_k)| \right). \quad (4.22)$$

It now follows from (3.8) that

$$|\bar{D}(\mathbf{a}_k^*; \mathbf{z})| = |D(\mathbf{a}_k^*; \mathbf{z})| + \sum_{m=1}^{n''} \frac{(i\gamma)^m}{m!} \sum_{k_1, \dots, k_m=1}^n |D(\mathbf{a}_k^*, \mathbf{a}_{k_1}^*, \dots, \mathbf{a}_{k_m}^*; \mathbf{z}, \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_m})|, \quad (4.23a)$$

$$|\bar{D}(\mathbf{z}^*; \mathbf{a}_k)| = |D(\mathbf{z}^*; \mathbf{a}_k)| + \sum_{m=1}^{n''} \frac{(i\gamma)^m}{m!} \sum_{k_1, \dots, k_m=1}^n |D(\mathbf{z}^*, \mathbf{a}_{k_1}^*, \dots, \mathbf{a}_{k_m}^*; \mathbf{a}_k, \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_m})|, \quad (4.23b)$$

where $n'' = \min(n-1, N-1)$. Referring to the expansion formulas (3.9) and (3.10), one has

$$|D(\mathbf{a}_k^*, \mathbf{a}_{k_1}^*, \dots, \mathbf{a}_{k_m}^*; \mathbf{z}, \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_m})| = |D|^{-1} |D(\mathbf{a}_k^*; \mathbf{z})| |D(\mathbf{a}_{k_1}^*, \dots, \mathbf{a}_{k_m}^*; \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_m})| \\ + |D|^{-1} \sum_{l=1}^m (-1)^l |D(\mathbf{a}_{k_l}^*; \mathbf{z})| |D(\mathbf{a}_k^*, \mathbf{a}_{k_1}^*, \dots, \mathbf{a}_{k_{l-1}}^*, \mathbf{a}_{k_{l+1}}^*, \dots, \mathbf{a}_{k_m}^*; \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_m})|, \quad (4.24a)$$

$$|D(\mathbf{z}^*, \mathbf{a}_{k_1}^*, \dots, \mathbf{a}_{k_m}^*; \mathbf{a}_k, \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_m})| = |D|^{-1} |D(\mathbf{z}^*; \mathbf{a}_k)| |D(\mathbf{a}_{k_1}^*, \dots, \mathbf{a}_{k_m}^*; \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_m})| \\ + |D|^{-1} \sum_{l=1}^m (-1)^l |D(\mathbf{z}^*; \mathbf{a}_{k_l})| |D(\mathbf{a}_{k_1}^*, \dots, \mathbf{a}_{k_m}^*; \mathbf{a}_k, \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_{l-1}}, \mathbf{a}_{k_{l+1}}, \dots, \mathbf{a}_{k_m})|. \quad (4.24b)$$

By introducing (4.23) into (4.22) and then using (4.24), P_2 takes the form

$$P_2 = \frac{i\gamma}{2|D|} \sum_{m=1}^{n''} \frac{(i\gamma)^m}{m!} \sum_{l=1}^m (-1)^l \times \\ \times \sum_{k, k_1, \dots, k_m=1}^n \left[-|D(\mathbf{a}_{k_l}^*; \mathbf{z})| |D(\mathbf{z}^*; \mathbf{a}_k)| |D(\mathbf{a}_k^*, \mathbf{a}_{k_1}^*, \dots, \mathbf{a}_{k_{l-1}}^*, \mathbf{a}_{k_{l+1}}^*, \dots, \mathbf{a}_{k_m}^*; \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_m})| \right. \\ \left. + |D(\mathbf{a}_k^*; \mathbf{z})| |D(\mathbf{z}^*; \mathbf{a}_{k_l})| |D(\mathbf{a}_{k_1}^*, \dots, \mathbf{a}_{k_m}^*; \mathbf{a}_k, \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_{l-1}}, \mathbf{a}_{k_{l+1}}, \dots, \mathbf{a}_{k_m})| \right]. \quad (4.25)$$

Interchange the indices k and k_l in the first term and then shift the row vector $\mathbf{a}_{k_l}^*$ in front of $\mathbf{a}_{k_{l+1}}$ and the column vector \mathbf{a}_k in front of \mathbf{a}_{k_1} , respectively. This leads to the following relation

$$|D(\mathbf{a}_k^*, \mathbf{a}_{k_1}^*, \dots, \mathbf{a}_{k_{l-1}}^*, \mathbf{a}_{k_{l+1}}^*, \dots, \mathbf{a}_{k_m}^*; \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_m})| \\ \rightarrow |D(\mathbf{a}_{k_l}^*, \mathbf{a}_{k_1}^*, \dots, \mathbf{a}_{k_{l-1}}^*, \mathbf{a}_{k_{l+1}}^*, \dots, \mathbf{a}_{k_m}^*; \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_{l-1}}, \mathbf{a}_k, \mathbf{a}_{k_{l+1}}, \dots, \mathbf{a}_{k_m})| \\ = |D(\mathbf{a}_{k_1}^*, \dots, \mathbf{a}_{k_m}^*; \mathbf{a}_k, \mathbf{a}_{k_1}, \dots, \mathbf{a}_{k_{l-1}}, \mathbf{a}_{k_{l+1}}, \dots, \mathbf{a}_{k_m})|.$$

Note that the value of the determinant is not altered since the total signature caused by the above manipulation is $(-1)^{2(l-1)} = 1$. Thus, the first term on the right-hand side of (4.25) coincides with the second term and cosequently, $P_2 = 0$. \square

4.3. Proof of (2.6)

Instead of proving (2.6) directly, we differentiate (2.5) by x and add the resultant expression to (2.6) and then prove the equation $P_3 = 0$, where

$$P_3 = f_{xx}f^* - f_x f_x^* - \frac{i\gamma}{2} \sum_{k=1}^n g_{k,x} g_k^*. \quad (4.26)$$

This reduces the total amount of calculations considerably and the proof becomes transparent. It now follows from (4.1), (4.3), (4.4), (4.6) and (4.8)-(4.10) that

$$P_3 = -\frac{1}{2} \{ |D(\mathbf{z}^*; \mathbf{z}_x)| + |D(\mathbf{z}_x^*; \mathbf{z})| \} |\bar{D}| - \frac{1}{4} |D(\mathbf{z}^*; \mathbf{z})| |\bar{D}(\mathbf{z}^*; \mathbf{z})| + \frac{i\gamma}{2} \sum_{k=1}^n |D(\mathbf{a}_k^*; \mathbf{z}_x)| |\bar{D}(\mathbf{z}^*; \mathbf{a}_k)|. \quad (4.27)$$

Differentiation of (4.15) with respect to x gives

$$|D(\mathbf{z}^*; \mathbf{z}_x)| + |D(\mathbf{z}_x^*; \mathbf{z})| = -i\gamma \sum_{k=1}^n |D(\mathbf{b}_k^*, \mathbf{z}^*; \mathbf{a}_k, \mathbf{z})|. \quad (4.28)$$

Inserting (4.15) and (4.28) into (4.27), P_3 can be put into the form

$$P_3 = \frac{i\gamma}{2} \sum_{k=1}^n \left\{ |\bar{D}| |D(\mathbf{b}_k^*, \mathbf{z}^*; \mathbf{a}_k, \mathbf{z})| + |D(\mathbf{z}^*; \mathbf{z})| |\bar{D}(\mathbf{a}_k^*, \mathbf{b}_k)| + |D(\mathbf{a}_k^*; \mathbf{z}_x)| |\bar{D}(\mathbf{z}^*; \mathbf{a}_k)| \right\}. \quad (4.29)$$

Note from (4.11), (4.13), (4.14) and Jacobi's identity (3.6) that

$$\begin{aligned} & |\bar{D}| |D(\mathbf{b}_k^*, \mathbf{z}^*; \mathbf{a}_k, \mathbf{z})| + |D(\mathbf{z}^*; \mathbf{z})| |\bar{D}(\mathbf{a}_k^*, \mathbf{b}_k)| \\ &= -|D(\mathbf{z}^*; \mathbf{a}_k)| \left\{ |D(\mathbf{b}_k^*; \mathbf{z})| + \frac{1}{2} |D(\mathbf{b}_k^*, \mathbf{z}^*; \mathbf{z}, \tilde{\mathbf{z}})| \right\} \\ &= -|D(\mathbf{z}^*; \mathbf{a}_k)| |\bar{D}(\mathbf{a}_k^*; \mathbf{z}_x)|. \end{aligned} \quad (4.30)$$

After substituting (4.30) into (4.29), P_3 becomes

$$P_3 = \frac{i\gamma}{2} \sum_{k=1}^n \left\{ -|\bar{D}(\mathbf{a}_k^*; \mathbf{z}_x)| |D(\mathbf{z}^*; \mathbf{a}_k)| + |D(\mathbf{a}_k^*; \mathbf{z}_x)| |\bar{D}(\mathbf{z}^*; \mathbf{a}_k)| \right\}. \quad (4.31)$$

This expression reduces to (4.22) if one replaces \mathbf{z}_x by \mathbf{z} . Hence, the proof of the relation $P_3 = 0$ completely parallels that of $P_2 = 0$ with P_2 from (4.22). \square

5. Alternative expression of the bright N -soliton solution

Here, we present an alternative expression of the bright N -soliton solution in terms of the determinants with smaller sizes when compared with those given by (2.13). Explicitly, we write it as a theorem:

Theorem 5.1. *The determinants f' and g'_j ($j = 1, 2, \dots, n$) given below satisfy the system of bilinear equations (2.4)-(2.6):*

$$f' = |A' + B'|, \quad g'_j = \begin{vmatrix} A' + B' & \mathbf{y}^T \\ -\mathbf{a}'_j & 0 \end{vmatrix}, \quad (j = 1, 2, \dots, n), \quad (5.1)$$

where A' and B' are $N \times N$ matrices and \mathbf{y} and \mathbf{a}'_j are N -component row vectors defined below:

$$A' = (a'_{jk})_{1 \leq j, k \leq N}, \quad a'_{jk} = \frac{1}{2} \frac{y_j y_k^*}{q_j + q_k^*}, \quad y_j = \exp(q_j x + i q_j^2 t), \quad (5.2a)$$

$$B' = (b'_{jk})_{1 \leq j, k \leq N}, \quad b'_{jk} = \frac{(\mu - i \gamma q_k^*) c'_{jk}}{q_j + q_k^*}, \quad c'_{jk} = \sum_{s=1}^n \alpha'_{sj} \alpha'_{sk}{}^*, \quad (5.2b)$$

$$\mathbf{y} = (y_1, y_2, \dots, y_N), \quad \mathbf{a}'_j = (\alpha'_{j1}, \alpha'_{j2}, \dots, \alpha'_{jN}). \quad (5.2c)$$

Here, q_j ($j = 1, 2, \dots, N$) and α'_{sj} ($s = 1, 2, \dots, n; j = 1, 2, \dots, N$) are complex parameters characterizing the solution.

Proof. The proof of the solution can be performed in the same way as that of (2.13) with (2.14). Indeed, the proof of (2.4), (2.5) and (2.6) reduce respectively to the relations (4.18), (4.22) and (4.31) in which the matrix D may be replaced simply by the matrix $A' + B'$. \square

Let us show that the determinants f and g_j from (2.13) are closely related to the determinants f' and g'_j given by (5.1). The following lemma is useful for this purpose:

Lemma 5.1. *The determinants f and g_j given by (2.13) can be rewritten in the form*

$$f = |I + AB|, \quad g_j = \begin{vmatrix} I + AB & \mathbf{z}^T \\ -\mathbf{a}_j^* & 0 \end{vmatrix}, \quad (j = 1, 2, \dots, n). \quad (5.3)$$

Proof. Multiplying f from (2.13) by a factor $\begin{vmatrix} B & -I \\ I & O \end{vmatrix}$ and performing the operation of matrix multiplication, the first expression of (5.3) follows immediately. Similarly, the second expression is obtained if one multiplies g_j by a factor $\begin{vmatrix} B & -I & \mathbf{0}^T \\ I & O & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & 1 \end{vmatrix}$. Indeed,

$$\begin{vmatrix} A & I \\ -I & B \end{vmatrix} \begin{vmatrix} B & -I \\ I & O \end{vmatrix} = \begin{vmatrix} I + AB & -A \\ O & I \end{vmatrix} = |I + AB|,$$

$$\begin{vmatrix} A & I & \mathbf{z}^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{a}_j^* & 0 \end{vmatrix} \begin{vmatrix} B & -I & \mathbf{0}^T \\ I & O & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & 1 \end{vmatrix} = \begin{vmatrix} I + AB & -A & \mathbf{z}^T \\ O & I & \mathbf{0}^T \\ -\mathbf{a}_j^* & \mathbf{0} & 0 \end{vmatrix} = \begin{vmatrix} I + AB & \mathbf{z}^T \\ -\mathbf{a}_j^* & 0 \end{vmatrix}.$$

Since the value of each factor multiplied from the right is 1, (5.3) follows. \square

We now establish the following theorem:

Theorem 5.2. *Under the parameterization $q_j = -p_j^*$ ($j = 1, 2, \dots, N$) and $\alpha'_{sj} = -\alpha_{sj}/(2c_j^*)$ ($s = 1, 2, \dots, n; j = 1, 2, \dots, N$), the determinants f, f', g_j and g'_j satisfy the relations*

$$f = c|A|f', \quad (5.4)$$

$$g_j = c|A|g'_j, \quad (j = 1, 2, \dots, n), \quad (5.5)$$

where

$$c = (-1)^N \prod_{l=1}^N (4c_l^* c_l), \quad c_l = \frac{\prod_{m=1}^N (p_l + p_m^*)}{\prod_{\substack{m=1 \\ m \neq l}}^N (p_l - p_m)}, \quad (l = 1, 2, \dots, N). \quad (5.6)$$

The parameters p_j ($j = 1, 2, \dots, N$) are assumed to satisfy the conditions $p_l + p_m^* \neq 0$ for all l and m and $p_l \neq p_m$ for $l \neq m$.

Proof. Let \tilde{A} be a Cauchy matrix of the form $\tilde{A} = \left(\frac{1}{2} \frac{1}{p_j + p_k^*} \right)$. Then, $A = (z_j \delta_{jk}) \tilde{A} (z_j^* \delta_{jk})$. Since $|\tilde{A}| \neq 0$ by virtue of the well-known formula for $|\tilde{A}|$ and the conditions imposed on the parameters p_j ($j = 1, 2, \dots, N$), the inverse of \tilde{A} exists, implying that A^{-1} exists

as well. Actually, it reads $A^{-1} = (z_j^* \delta_{jk})^{-1} \tilde{A}^{-1} (z_j \delta_{jk})^{-1}$. Using the explicit expression of \tilde{A}^{-1} , i.e., $\left(\frac{2c_j^* c_k}{p_j^* + p_k}\right)$ [11], the inverse matrix A^{-1} can be written in the form

$$A^{-1} = \left(\frac{2c_j^* c_k}{p_j^* + p_k} \frac{1}{z_j^* z_k} \right). \quad (5.7)$$

Applying the basic properties of determinants to f and g_j from (5.3) gives

$$f = |A| |A^{-1} + B|, \quad (5.8)$$

$$g_j = |A| \begin{vmatrix} A^{-1} + B & A^{-1} \mathbf{z}^T \\ -\mathbf{a}_j^* & 0 \end{vmatrix}, \quad (j = 1, 2, \dots, n). \quad (5.9)$$

The j th element of the column vector $A^{-1} \mathbf{z}^T$ is

$$\begin{aligned} (A^{-1} \mathbf{z}^T)_j &= \sum_{l=1}^N (A^{-1})_{jl} z_l \\ &= \frac{2c_j^*}{z_j^*} \sum_{l=1}^N \frac{1}{p_j^* + p_l} \frac{\prod_{m=1}^N (p_l + p_m^*)}{\prod_{\substack{m=1 \\ m \neq l}}^N (p_l - p_m)} \\ &= \frac{2c_j^*}{z_j^*} \sum_{l=1}^N \frac{\prod_{\substack{m=1 \\ m \neq j}}^N (p_l + p_m^*)}{\prod_{\substack{m=1 \\ m \neq l}}^N (p_l - p_m)}. \end{aligned} \quad (5.10)$$

By Euler's formula, the sum in the last line turns out to be 1 and hence $(A^{-1} \mathbf{z}^T)_j = 2c_j^*/z_j^*$.

Introducing this relation into g_j ,

$$g_j = |A| \begin{vmatrix} A^{-1} + B & \hat{\mathbf{z}}^T \\ -\mathbf{a}_j^* & 0 \end{vmatrix}, \quad (j = 1, 2, \dots, n), \quad (5.11)$$

where $\hat{\mathbf{z}} = (2c_1^*/z_1^*, 2c_2^*/z_2^*, \dots, 2c_N^*/z_N^*)$ is an N -component row vector.

The next step is to modify the determinants f' and g'_j . By means of the parametrization $q_j = -p_j^*$, y_j from (5.2a) is related to z_j from (2.14a) by the relation $y_j = z_j^{*-1}$. Similarly, the relation $c'_{jk} = c_{jk}/(4c_j^* c_k)$ follows from (2.14b) and (5.2b) and the parametrization $\alpha'_{sj} = -\alpha_{sj}/(2c_j^*)$. Substitution of these relations into f' gives

$$\begin{aligned} f' &= \left| \left(\frac{1}{2} \frac{y_j y_k^*}{q_j + q_k^*} + \frac{(\mu - i\gamma q_k^*) c'_{jk}}{q_j + q_k^*} \right) \right| \\ &= \frac{(-1)^N}{\prod_{l=1}^N (4c_l^* c_l)} \left| \left(\frac{2c_j^* c_k}{p_j^* + p_k} \frac{1}{z_j^* z_k} + \frac{(\mu + i\gamma p_k) c_{jk}}{p_j^* + p_k} \right) \right| \\ &= c^{-1} |A^{-1} + B|, \end{aligned} \quad (5.12)$$

where in passing to the second line, the factor $1/(2c_j^*)$ has been extracted from the j th row ($j = 1, 2, \dots, N$) and the factor $-1/(2c_k)$ from the k th column ($k = 1, 2, \dots, N$), respectively. The similar procedure applied to g'_j leads to the expression

$$g'_j = c^{-1} \begin{vmatrix} A^{-1} + B & \hat{\mathbf{z}}^T \\ -\mathbf{a}_j^* & 0 \end{vmatrix}, \quad (j = 1, 2, \dots, n). \quad (5.13)$$

The relation (5.4) follows from (5.8) and (5.12) whereas the relation (5.5) follows from (5.11) and (5.13). \square

Thus, we have obtained the two different expressions for the bright N -soliton solution of the system of nonlinear PDEs (2.2). Explicitly, they read $u_j = g_j/f = g'_j/f'$ ($j = 1, 2, \dots, n$).

The following proposition provides an alternative proof of theorem 5.1:

Proposition 5.1. *If f and g_j given respectively by (5.4) and (5.5) satisfy the system of bilinear equations (2.4)-(2.6), then f' and g'_j satisfy the same system of equations, and vice versa.*

Proof. Substituting (5.4) and (5.5) into (2.4) and using the definition of the bilinear operators,

$$c^2 |A|^2 (iD_t g'_j \cdot f' + D_x^2 g'_j \cdot f') + c^2 (D_x^2 |A| \cdot |A|) g'_j f' = 0. \quad (5.14)$$

The Cauchy type determinant $|A|$ can be modified, after extracting the factor z_j from j th row ($j = 1, 2, \dots, N$) and the factor z_k^* from k th column ($k = 1, 2, \dots, N$), respectively, in the form

$$\begin{aligned} |A| &= \prod_{l=1}^N (z_l z_l^*) \left| \left(\frac{1}{2} \frac{1}{p_j + p_k^*} \right) \right| \\ &= \exp \left[\sum_{l=1}^N (p_l + p_l^*) x + i \sum_{l=1}^N (p_l^2 - p_l^{*2}) t \right] \left| \left(\frac{1}{2} \frac{1}{p_j + p_k^*} \right) \right|. \end{aligned} \quad (5.15)$$

Differentiation of $|A|$ with respect to x gives

$$|A|_x = \sum_{l=1}^N (p_l + p_l^*) |A|, \quad |A|_{xx} = \left\{ \sum_{l=1}^N (p_l + p_l^*) \right\}^2 |A|. \quad (5.16)$$

It immediately follows from (5.16) that

$$D_x^2 |A| \cdot |A| = 2(|A||A|_{xx} - |A|_x^2) = 0. \quad (5.17)$$

It is seen from (5.14), (5.17) and the relation $c|A| \neq 0$ that f' and g'_j satisfy the bilinear equation (2.4). The remaining part of the proposition can be proved in the same way if one uses (5.17) and the reality of $|A|$, i.e., $|A|^* = |A|$ which is a consequence of the Hermitian nature of the matrix A . The proof of the converse proposition proceeds in the same way if one uses the relation $D_x^2 |A|^{-1} \cdot |A|^{-1} = 0$ in place of (5.17). \square

6. A continuum model

The n -component system (1.1) yields a continuum model when one takes a limit $n \rightarrow \infty$. It represents a (2+1)-dimensional nonlocal modified NLS equation of the form

$$i q_t + q_{xx} + \mu \left(\int_{-\infty}^{\infty} |q|^2 dy \right) q + i\gamma \left(\int_{-\infty}^{\infty} |q|^2 dy q \right)_x = 0, \quad q = q(x, y, t). \quad (6.1)$$

Recall that when $\gamma = 0$, this equation reduces to a (2+1)-dimensional nonlocal NLS equation proposed by Zakharov [12]. The exact method of solution for equation (6.1) can be developed following the same procedure as that for the system of nonlinear PDEs (1.1). Hence, we summarize the main results.

First, application of the gauge transformation

$$q = u \exp \left[-\frac{i\gamma}{2} \int_{-\infty}^x \int_{-\infty}^{\infty} |u(x, y, t)|^2 dx dy \right], \quad u = u(x, y, t), \quad (6.2)$$

to the system (6.1) subjected to the the boundary conditions $q \rightarrow 0, u \rightarrow 0$ $|x| \rightarrow \infty$ transforms (6.1) to a nonlocal nonlinear PDE for u

$$i u_t + u_{xx} + \mu \left(\int_{-\infty}^{\infty} |u|^2 dy \right) u + i\gamma \left(\int_{-\infty}^{\infty} u^* u_x dy \right) u = 0. \quad (6.3)$$

The proposition below is an analog of proposition 2.1:

Proposition 6.1 *By means of the dependent variable transformation*

$$u = \frac{g}{f}, \quad (6.4)$$

equation (6.3) can be decoupled into the following system of bilinear equations for $f = f(x, t)$ and $g = g(x, y, t)$

$$(iD_t + D_x^2)g \cdot f = 0, \quad (6.5)$$

$$D_x f \cdot f^* = \frac{i\gamma}{2} \int_{-\infty}^{\infty} |g|^2 dy, \quad (6.6)$$

$$D_x^2 f \cdot f^* = \mu \int_{-\infty}^{\infty} |g|^2 dy + \frac{i\gamma}{2} \int_{-\infty}^{\infty} D_x g \cdot g^* dy. \quad (6.7)$$

Proof. The proof proceeds exactly as that of proposition 2.1. Formally, one may simply replace the sum $\sum_{k=1}^n$ by the integral $\int_{-\infty}^{\infty} dy$. \square

It follows from (6.2), (6.4) and (6.6) that

$$q = \frac{gf^*}{f^2}, \quad (6.8)$$

which is just a continuum limit of (2.13).

The following theorem can be derived from a continuum limit of the bright N -soliton solution given by theorem 2.1 and theorem 5.1:

Theorem 6.1. *The system of bilinear equations (6.5)-(6.7) admits the following two different expressions f, g and f', g' for the bright N -soliton solution :*

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}, \quad g = \begin{vmatrix} A & I & \mathbf{z}^T \\ -I & B & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{a}^* & 0 \end{vmatrix}, \quad (6.9)$$

$$f' = |A' + B'|, \quad g' = \begin{vmatrix} A' + B' & \mathbf{y}^T \\ -\mathbf{a}'^* & 0 \end{vmatrix}. \quad (6.10)$$

Here, A and B are $N \times N$ matrices given respectively by (2.14a) and (2.14b) with c_{jk} being replaced by $\int_{-\infty}^{\infty} \alpha_j(y) \alpha_k^*(y) dy$, A' and B' are $N \times N$ matrices given respectively by (5.2a) and (5.2b) with c'_{jk} being replaced by $\int_{-\infty}^{\infty} \alpha'_j(y) \alpha'_k{}^*(y) dy$ and $\mathbf{a} = \mathbf{a}(y) = (\alpha_1, \alpha_2, \dots, \alpha_N)$ and $\mathbf{a}' = \mathbf{a}'(y) = (\alpha'_1, \alpha'_2, \dots, \alpha'_N)$ are N -component row vectors where α_j and α'_j ($j = 1, 2, \dots, N$) are continuous functions of y .

Proof. The proof can be done in the same way as that of theorem 2.1 and theorem 5.1.

□

Theorem 6.2. *Under the parameterization $q_j = -p_j^*$ and $\alpha'_j = -\alpha_j/(2c_j^*)$ ($j = 1, 2, \dots, N$), the determinants f, f', g and g' satisfy the relations*

$$f = c|A|f', \quad (6.11)$$

$$g = c|A|g', \quad (6.12)$$

where c is defined by (5.6) and the parameters p_j ($j = 1, 2, \dots, N$) are specified such that $p_l + p_m^* \neq 0$ for all l and m and $p_l \neq p_m$ for $l \neq m$.

Proof. The proof parallels theorem 5.2. □

Proposition 6.2. *If f and g given by (6.9) satisfy the system of bilinear equations (6.5)-(6.7), then f' and g' given by (6.11) and (6.12) satisfy the same system of equations, and vice versa.*

Proof. The proof is completely parallel to that of proposition 5.1. □

7. Concluding remarks

In this paper, we have obtained the two different expressions for the bright N -soliton solution of an n -component modified NLS equation and found a simple relationship between them. We also have presented the bright N -soliton solution of a continuum model arising from the system as the number n of the dependent variables tends to infinity. These solutions include, as special cases, existing solutions for a multi-component NLS equation. Actually, when $\gamma = 0$, the system of nonlinear PDEs (1.1) reduces to an n -component NLS equation. It admits the bright N -soliton solution of the form $q_j = g_j/f = g'_j/f'$ ($j = 1, 2, \dots, n$). This fact follows from (2.12) and the relations $f^* = f$ and $f'^* = f'$. See (2.13) and (5.1). The solution (2.13) with $\gamma = 0$ has been obtained in [13] using a direct method whereas the solution (5.3) with $\gamma = 0$ has been constructed by means of the IST [14]. An alternative expression (5.1) of the solution with $\gamma = 0$ has

been derived by employing a method of algebraic geometry [15]. For a continuum model (6.1) with $\gamma = 0$, the solution takes the form $q = g/f$. The bright N -soliton solution (6.9) with $\gamma = 0$ has been found in [16] by a direct method.

In a future work, we will investigate the various features of the multi-component bright solitons. In particular, we will be concerned with the effect of the parameters μ and γ which characterize the different types of nonlinearities on the interaction process of solitons.

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